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THERMOELASTIC POSTBUCKLING RESPONSE OF STRIP DELAMINATION MODELS

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Abstract—Postbuckling solutions are obtained for multilayered strip delamination models subjected to in-plane compression, bidirectional bending, twisting, and a temperature load that may vary arbitrarily in the thickness direction. The differential equations of equilibrium, the end conditions and the continuity conditions at the junction of the sublaminates are reduced to a system of algebraic equations governing the deformation parameters. Besides providing closed-form postbuckling solutions, these equations also reveal explicitly the effects on buckling due to various factors including delamination geometry, anisotropic elastic and thermal expansion coefficients, in-plane force and strain loads, bending and twisting curvatures, and the temperature field. A simple expression is given for the energy release rate in terms of the mid-plane strains and the curvatures of the sublaminates at the delamination front. A moderate temperature gradient in the thickness direction may severely aggravate the postbuckling deformation and increase the energy release rate. © 1998 Elsevier Science Ltd. All rights reserved.

NOMENCLATURE

x, y, z	Cartesian coordinates referred to the center of the laminate
z'	thickness coordinate referred to the middle plane of a sublaminate
$\sigma_{\beta\gamma}$ $(\beta,\gamma = 1,2)$	in-plane components of the stress
$T(z)$	temperature load
T_u, T_l	temperature on the top and bottom surfaces of the laminate
$\mathbf{Q}^{(k)}$	in-plane stiffness matrix of the kth layer
$\alpha^{(k)}_{\beta\gamma}$ $(\beta,\gamma = 1,2)$	in-plane thermal expansion coefficients of the kth layer
$t, H, h \equiv t-H$	thickness of the intact and disbonded sublaminates
$2L$, $2a$	length of the laminate and of the delamination
$b \equiv L-a$	
$\mathbf{A} \equiv [A_{ij}]$	extensional stiffness matrix
$\mathbf{B} \equiv [B_{ij}]$	bending-extension coupling matrix
$\mathbf{D} \equiv [D_{ij}]$	bending and twisting stiffness matrix
$[\underline{A}_{ij}]$, $[\underline{B}_{ij}]$, $[\underline{D}_{ij}]$	stiffness matrices of the lower disbonded sublaminate
$[\bar{A}_{ij}]$, $[\bar{B}_{ij}]$, $[\bar{D}_{ij}]$	stiffness matrices of the upper disbonded sublaminate
Δ, $\underline{\Delta}$, $\bar{\Delta}$	sublaminate properties defined by eqns (11), (17), etc.
D, $\underline{D}$, $\bar{D}$	sublaminate properties defined by eqns (12), (14), etc.
$\varepsilon^0_{\beta\gamma}$, $\underline{\varepsilon}^0_{\beta\gamma}$, $\bar{\varepsilon}^0_{\beta\gamma}$	middle plane strains of the sublaminates
$\kappa_{\beta\gamma}$, $\underline{\kappa}_{\beta\gamma}$, $\bar{\kappa}_{\beta\gamma}$	middle plane curvatures of the sublaminates
κ^0_x, κ_y, κ_{xy}	curvature loads imposed on the delamination model
$N_{\beta\gamma}$, $\underline{N}_{\beta\gamma}$, $\bar{N}_{\beta\gamma}$	in-plane normal and shearing forces
$M_{\beta\gamma}$, $\underline{M}_{\beta\gamma}$, $\bar{M}_{\beta\gamma}$	bending and twisting moments
$N^*_{\beta\gamma}$, $\underline{N}^*_{\beta\gamma}$, $\bar{N}^*_{\beta\gamma}$	thermal forces
$M^*_{\beta\gamma}$, $\underline{M}^*_{\beta\gamma}$, $\bar{M}^*_{\beta\gamma}$	thermal moments
F_x, F_{xy}, $\underline{F}_x$, $\underline{F}_{xy}$, $\bar{F}_x$, $\bar{F}_{xy}$	effective forces defined by eqns (13), (19) and (20)
w, $\underline{w}$, $\bar{w}$	deflections
ε, γ, ξ, η, λ	deformation parameters of the intact sublaminate [eqn (3a,b,c)]
$\underline{\varepsilon}$, $\underline{\gamma}$, $\underline{\xi}$, $\underline{\eta}$, $\underline{\lambda}$	deformation parameters of the lower sublaminate [eqn (3d,e,f)]
$\bar{\varepsilon}$, $\bar{\gamma}$, $\bar{\xi}$, $\bar{\eta}$, $\bar{\lambda}$	deformation parameters of the upper sublaminate [eqn (3g,h,i)]

$P \equiv -N_x = D(\lambda t)^2$, $\underline{P} \equiv -\underline{N}_x = \underline{D}(\underline{\lambda} t)^2$, $\bar{P} \equiv -\bar{N}_x = \bar{D}(\bar{\lambda} t)^2$, $S \equiv N_{xy}$

$\mathscr{P} \equiv P(L/t)^2/(\pi^2 D)$	normalized axial compression load
G	strain energy release rate

1. INTRODUCTION

Buckling and postbuckling behavior of composite beams and plates with internal delaminations have been the object of intensive research in the past decade (Garg, 1988; Storakers, 1989). Although hygrothermal loads, in addition to mechanical loads, may aggravate the instability of composite laminates with interfacial defects, their effects on the delamination problem have not been systematically investigated. It was shown in a recent analytical study that a delaminated thin layer on the hot surface of a non-uniformly heated plate may be particularly vulnerable to the initiation of local buckling (Yin, 1994). The study was based on the solutions of the linearized equilibrium equation for a multilayered strip delamination model subjected to a temperature load that may vary arbitrarily in the thickness direction. The simplicity of the formulation (using laminated plate theory and von Karman's strain–displacement relations) and of the resulting analytical expressions allows the effects of anisotropic stiffness parameters, thermal expansion coefficients and mechanical and thermal loads to be separately examined. One finds that drastic reduction in the bifurcation loads may result from a temperature gradient in the thickness direction, and that bending-stretching coupling of the sublaminate stiffness contributes significantly to the instability of the model.

As pointed out in an earlier study, a composite laminate with a shallow delamination may buckle in a local mode under a relatively small compression load (Yin *et al.*, 1986). Large deflection with severe peeling and shearing actions near the crack front may not arise until the axial load increases significantly beyond its critical value at bifurcation. Therefore, the bifurcation load is usually not an indication of the load carrying capacity of the delaminated plate. An assessment of the detrimental effects of delamination damage and crack growth requires the solution of nonlinear governing equations in a relatively advanced postbuckling stage.

In the following section, the kinematical formulation of a previous paper (Yin, 1994) for cylindrical buckling of laminated strip delamination models will be extended to accommodate bidirectional bending and twisting of the sublaminates. This extension is required because even a uniform temperature load may cause bending and twisting in an unbalanced laminate, so that the usual assumption of cylindrical (plane strain) deformation is generally not valid. While the previous study leads to an eigenvalue problem and a characteristic equation associated with the linearized problem of bifurcation, the present analysis reduces the thermoelastic postbuckling problem to a system of three algebraic equations governing a reduced set of deformation parameters, i.e., eqns (23)–(25) for the three parameters θ, $\underline{\lambda}$ and $\bar{\lambda}$. Besides providing exact postbuckling solutions based on the nonlinear strain–displacement relation of the von Karman plate theory, these equations also allow the effects of all anisotropic sublaminate stiffness matrices, the thermal expansion coefficients and the thermal and mechanical loads to be examined separately, or in combination, in connection with the postbuckling deformation.

Solutions are computed for several delamination models (homogeneous and isotropic; laminated with cross-ply or angle-ply configurations) with various combinations of the length and depth of delamination and mechanical and temperature loads. The energy release rate associated with postbuckling delamination growth is evaluated by using an exact general expression in terms of the sublaminate strains and curvatures at the delamination front.

2. GENERALIZED 2-D DEFORMATION OF THE STRIP DELAMINATION MODEL

Consider a laminated beam-plate of thickness t and axial length $2L$ containing an across-the-width delamination of length $2a$ at a depth h beneath the upper surface. The delamination is assumed to be located symmetrically with respect to the two clamped ends of the laminate (Fig. 1). Let $[A_{ij}]$, $[B_{ij}]$ and $[D_{ij}]$ denote the stiffness matrices of the intact segment, and let $[\underline{A}_{ij}]$, $[\underline{B}_{ij}]$ and $[\underline{D}_{ij}]$ and $[\bar{A}_{ij}]$, $[\bar{B}_{ij}]$ and $[\bar{D}_{ij}]$ stand, respectively, for the corresponding matrices of the lower and upper delaminated sublaminates. The following equalities are easily established (where $H = t-h$):

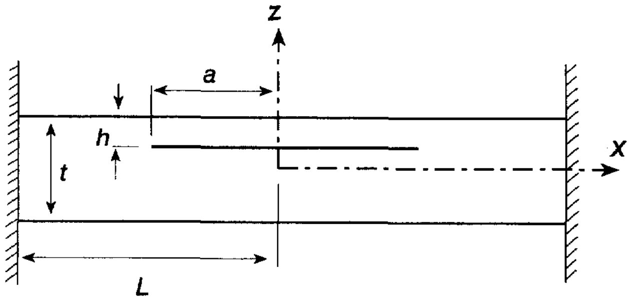

Fig. 1. One-dimensional delamination model.

$$A_{ij} = \underline{A}_{ij} + \bar{A}_{ij}, \quad B_{ij} = \underline{B}_{ij} + \bar{B}_{ij} + \underline{A}_{ij}h/2 - \bar{A}_{ij}H/2,$$

$$D_{ij} = \underline{D}_{ij} + \bar{D}_{ij} + \underline{B}_{ij}h - \bar{B}_{ij}H + \underline{A}_{ij}h^2/4 + \bar{A}_{ij}H^2/4. \tag{1}$$

The delamination model is subjected to an axial compressive load $-N_x = P$, an in-plane shearing force $N_{xy} = S$ and a bending moment $M_x = M$ at its two ends, $x = L$. In addition, a temperature load $T(z)$ is applied. The temperature load, by itself or in conjunction with mechanical edge loads, generally causes extension, bending and twisting of the anisotropic laminate. Hence, besides P, S and M, three additional parameters are needed to completely specify the state of loading of the model. These load parameters are chosen to be the midplane strain ε_y^0 and the curvatures κ_y and κ_{xy}, rather than the conjugate forces and moments, N_y, M_y and M_{xy}. The reason for this choice is that, in the generalized 2-D solutions of the following analysis, ε_y^0, κ_y and κ_{xy} are constant parameters whereas N_y, M_y and M_{xy} are functions of x. For the same reason, we shall also replace the boundary moment M by a constant curvature parameter κ_x^0 as one of the specified load parameters in the formulation of the postbuckling problem.

We seek generalized 2-D postbuckling solutions of the strip delamination model, i.e., deformations in which the stress and strain fields are independent of the coordinate y. For such solutions, the intact and disbonded sublaminates in the right half of the delaminated plate undergo transverse deflections of the following forms:

$$w(x,y) = A\{\cos\lambda(L-x) - 1\} + \kappa_x^0 x^2/2 + \kappa_{xy}xy + \kappa_y y^2/2, \quad (a \leqslant x \leqslant L) \tag{2a}$$

$$\underline{w}(x,y) = A\{(\lambda\sin\lambda b/\underline{\lambda}\sin\underline{\lambda}a)(\cos\underline{\lambda}a - \cos\underline{\lambda}x) + \cos\lambda b - 1\} + \kappa_x^0 x^2/2 + \kappa_{xy}xy + \kappa_y y^2/2,$$

$$\bar{w}(x,y) = A\{(\lambda\sin\lambda b/\bar{\lambda}\sin\bar{\lambda}a)(\cos\bar{\lambda}a - \cos\bar{\lambda}x) + \cos\lambda b - 1\} + \kappa_x^0 x^2/2 + \kappa_{xy}xy + \kappa_y y^2/2,$$

$$(0 \leqslant x \leqslant a) \tag{2b,c}$$

where λ, $\underline{\lambda}$, $\bar{\lambda}$ and A are the constants yet to be determined, $b \equiv L-a$, and where x denotes the axial coordinate measured from the midpoint of the model. These expressions satisfy the continuity of deflection and slope at the crack tip and the symmetry conditions at $x = 0$. The tangential strains in the middle planes of the three sublaminates have the following forms:

$$\varepsilon_y^0 = \beta, \quad \varepsilon_x^0 = \varepsilon + \xi\cos\lambda(L-x), \quad \gamma_{xy}^0 = \gamma + \eta\cos\lambda(L-x), \quad (a \leqslant x \leqslant L) \tag{3a,b,c}$$

$$\underline{\varepsilon}_y^0 = \beta + \kappa_y h/2, \quad \underline{\varepsilon}_x^0 = \underline{\varepsilon} + \underline{\xi}\cos\underline{\lambda}x, \quad \underline{\gamma}_{xy}^0 = \underline{\gamma} + \underline{\eta}\cos\underline{\lambda}x, \quad (0 \leqslant x \leqslant a) \tag{3d,e,f}$$

$$\bar{\varepsilon}_y^0 = \beta - \kappa_y H/2, \quad \bar{\varepsilon}_x^0 = \bar{\varepsilon} + \bar{\xi}\cos\bar{\lambda}x, \quad \bar{\gamma}_{xy}^0 = \bar{\gamma} + \bar{\eta}\cos\bar{\lambda}x, \quad (0 \leqslant x \leqslant a). \tag{3g,h,i}$$

It will be shown that, with appropriate choices of constant parameters, the deformation expressed by eqns (2) and (3) yields stress and moment resultants that satisfy the equilibrium equations in each sublaminate [eqn (21) in Section 4].

We assume that the sublaminates deform according to the Kirchhoff–Love assumption of the classical plate theory. Then the strain field in each sublaminate is determined by the middle-plane strains and the curvatures

$$\kappa_x(x) \equiv w_{,xx}, \quad \underline{\kappa}_x(x) \equiv \underline{w}_{,xx}, \quad \bar{\kappa}_x(x) \equiv \bar{w}_{,xx},$$

$$w_{,yy} = \underline{w}_{,yy} = \bar{w}_{,yy} \equiv \kappa_y, \quad w_{,xy} = \underline{w}_{,xy} = \bar{w}_{,xy} \equiv \kappa_{xy},$$

(where the subscript variables following the commas indicate partial differentiation). Consequently, each generalized 2-D postbuckling deformation of the strip delamination model is completely characterized by the twenty deformation parameters A, λ, $\underline{\lambda}$, $\bar{\lambda}$, β, κ_x^0, κ_y, κ_{xy}, ε, γ, ξ, η, $\underline{\varepsilon}$, $\underline{\gamma}$, $\underline{\xi}$, $\underline{\eta}$, $\bar{\varepsilon}$, $\bar{\gamma}$, $\bar{\xi}$ and $\bar{\eta}$. These deformation parameters may be determined in terms of the temperature load $T(z)$ and the six specified mechanical load parameters β, κ_x^0, κ_y, κ_{xy}, P and S.

The imposed curvature loads $\kappa_{\beta\gamma}$ are considered to be infinitesimal in the sense that $\kappa_{\beta\gamma}L \ll t/L$. With this assumption, linear strain–displacement relations are valid in the prebuckling states [where the amplitude parameter A of eqn (2) vanishes], i.e., the nonlinear and non-constant terms $w_{,\beta}w_{,\gamma}/2$ are negligible and, therefore, do not affect the spatial constancy of the prebuckling membrane strains. In the postbuckling stage, the curvatures κ_x^0, κ_y and κ_{xy}, which are determined by the temperature load and the edge constraints, remain small compared to t/L^2, but the deflections associated with the amplitude parameter A generally yield significant rotations of the sublaminate midplanes with respect to the y-axis. Indeed, eqns (2a,b,c) imply that these rotations are of the order λA and, therefore, are comparable in magnitude to t/L since the amplitude A is comparable to t in an advanced stage of postbuckling deformation. Consequently, the nonlinear strain displacement relations in the postbuckling stage may be written as follows

$$\varepsilon_x^0 = \partial u/\partial x + (1/2)\{A\lambda \sin\lambda(L-x)\}^2, \quad \varepsilon_y^0 = \partial v/\partial y, \quad \gamma_{xy}^0 = \partial u/\partial y + \partial v/\partial x, \tag{4a}$$

$$\underline{\varepsilon}_x^0 = \partial \underline{u}/\partial x + (1/2)(A\lambda \sin\lambda b \sin\underline{\lambda}x/\sin\underline{\lambda}a)^2, \quad \underline{\varepsilon}_y^0 = \partial \underline{v}/\partial y, \quad \underline{\gamma}_{xy}^0 = \partial \underline{u}/\partial y + \partial \underline{v}/\partial x, \tag{4b}$$

$$\bar{\varepsilon}_x^0 = \partial \bar{u}/\partial x + (1/2)(A\lambda \sin\lambda b \sin\bar{\lambda}x/\sin\bar{\lambda}a)^2, \quad \bar{\varepsilon}_y^0 = \partial \bar{v}/\partial y, \quad \bar{\gamma}_{xy}^0 = \partial \bar{u}/\partial y + \partial \bar{v}/\partial x, \tag{4c}$$

where u, v, $\underline{u}$, $\underline{v}$, $\bar{u}$ and $\bar{v}$ denote the midplane displacements of the three sublaminates.

3. THERMOELASTIC CONSTITUTIVE EQUATIONS OF THE SUBLAMINATES

In the kth layer of the intact or disbonded sublaminate, the in-plane stresses σ_x, σ_y and τ_{xy} depend on the temperature load $T(z)$ and the middle-plane strains and curvatures according to the thermoelastic constitutive equation

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} - \mathbf{Q}^{(k)}\left(\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} - z'\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}\right) = -T(z)\mathbf{Q}^{(k)}\begin{Bmatrix}\alpha_x^{(k)}\\ \alpha_y^{(k)}\\ \alpha_{xy}^{(k)}\end{Bmatrix} \tag{5}$$

where $\mathbf{Q}^{(k)}$ is a 3×3 symmetric elastic matrix and where $\alpha_x^{(k)}$, $\alpha_y^{(k)}$ and $\alpha_{xy}^{(k)}$ are the thermal expansion coefficients of the anisotropic layer. In eqn (5), z' refers to the thickness coordinate measured from the middle plane of the sublaminate. Hence z is identical to z' in the intact sublaminate but differs from z' by amounts $H/2$ and $-h/2$, respectively, in the upper and lower disbonded sublaminates.

Integrating the in-plane stress components and their first moments (with respect to the middle plane of the sublaminate) across the thickness of the successive layers and summing the results over all layers of the sublaminate, one obtains the stress and moment resultants

$$N_x = \int \sigma_x \,\mathrm{d}z, \quad N_y = \int \sigma_y \,\mathrm{d}z, \quad N_{xy} = \int \tau_{xy} \,\mathrm{d}z,$$
$$M_x = -\int z'\sigma_x \,\mathrm{d}z, \quad M_y = -\int z'\sigma_y \,\mathrm{d}z, \quad M_{xy} = -\int z'\tau_{xy} \,\mathrm{d}z.$$

If σ_x, σ_y and τ_{xy} on the right-hand side of these expressions are replaced, respectively, by the first, second and third elements of the right-hand side of eqn (5), one obtains, instead, the thermal forces N_x^*, N_y^*, N_{xy}^* and the thermal moments M_x^*, M_y^*, M_{xy}^*. They stand for the constant forces and moments that would result from the temperature load $T(z)$ in the hypothetical state when the boundary constraints prevent any deformation of the delamination model. The definitions of the thermal forces and thermal moments for the three sublaminates imply the relations:

$$N_{\alpha\beta}^* = \underline{N}_{\alpha\beta}^* + \bar{N}_{\alpha\beta}^*, \quad M_{\alpha\beta}^* = \underline{M}_{\alpha\beta}^* + \bar{M}_{\alpha\beta}^* + (h/2)\underline{N}_{\alpha\beta}^* - (H/2)\bar{N}_{\alpha\beta}^* \quad (\alpha, \beta = 1, 2). \tag{6}$$

Furthermore, eqn (5) yields the following thermoelastic constitutive equation of the intact sublaminate

$$\begin{Bmatrix} N_x - N_x^* \\ N_y - N_y^* \\ N_{xy} - N_{xy}^* \\ M_x - M_x^* \\ M_y - M_y^* \\ M_{xy} - M_{xy}^* \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{7}$$

where

$$\mathbf{A} \equiv [A_{ij}] \equiv \Sigma \int \mathbf{Q}^{(k)} \,\mathrm{d}z', \quad \mathbf{B} \equiv [B_{ij}] \equiv -\Sigma \int z' \mathbf{Q}^{(k)} \,\mathrm{d}z',$$
$$\mathbf{D} \equiv [D_{ij}] \equiv \Sigma \int (z')^2 \mathbf{Q}^{(k)} \,\mathrm{d}z', \quad (i, j = 1, 2, 6)$$

and the summations extend over all layers. Constitutive relations of the same form are valid for the lower and upper disbonded sublaminates, provided that the various quantities in eqn (7) are replaced by corresponding quantities with underlines and overhead bars.

4. ALGEBRAIC EQUATIONS GOVERNING THE DEFORMATION PARAMETERS OF THE POSTBUCKLING SOLUTION

It has been shown that, in cylindrical deformation of the strip delamination model (where $\kappa_x^0 = \kappa_y = \kappa_{xy} = 0$), the equilibrium equations of each sublaminate determine the deformation parameters λ, ε, γ, ξ and η of eqns (2) and (3) in terms of N_x, N_{xy}, N_x^*, N_{xy}^*, β and the deflection amplitude A (see Yin, 1994). More general results may be obtained with the inclusion of the bending and twisting curvatures. In the intact segment, where $N_x \equiv -P$ and $N_{xy} \equiv S$, the new results accounting for κ_x^0, κ_y and κ_{xy} are given by

$$\lambda = (1/t)\sqrt{(P/D)}, \tag{8}$$

$$\varepsilon = (1/\Delta)\{A_{16}(-S + F_{xy}) - A_{66}(D\lambda^2 t^2 + F_x)\}, \tag{9a}$$

$$\gamma = -(1/\Delta)\{A_{11}(-S + F_{xy}) - A_{16}(D\lambda^2 t^2 + F_x)\}, \tag{9b}$$

$$\xi = (1/\Delta)(A_{66}B_{11}/t - A_{16}B_{16}/t)\lambda^2 tA, \tag{10a}$$

$$\eta = (1/\Delta)(A_{11}B_{16}/t - A_{16}B_{11}/t)\lambda^2 tA, \tag{10b}$$

where

$$\Delta \equiv A_{11}A_{66}-(A_{16})^2. \tag{11}$$

$$D \equiv (1/\Delta)\begin{vmatrix} A_{11} & A_{16} & B_{11}/t \\ A_{16} & A_{66} & B_{16}/t \\ B_{11}/t & B_{16}/t & D_{11}/t^2 \end{vmatrix} \tag{12}$$

and F_x and F_{xy} denote the following combinations of the thermal forces and the strain and curvature loads:

$$F_x \equiv N_x^* + A_{12}\beta + B_{11}\kappa_x^0 + B_{12}\kappa_y + B_{16}\kappa_{xy}, \tag{13a}$$

$$F_{xy} \equiv N_{xy}^* + A_{26}\beta + B_{16}\kappa_x^0 + B_{26}\kappa_y + B_{66}\kappa_{xy}. \tag{13b}$$

Results corresponding to eqns (8)–(13) are given for the lower sublaminate as follows

$$\underline{P}/(\underline{\lambda}t)^2 = (1/\underline{\Delta})\begin{vmatrix} \underline{A}_{11} & \underline{A}_{16} & \underline{B}_{11}/t \\ \underline{A}_{16} & \underline{A}_{66} & \underline{B}_{16}/t \\ \underline{B}_{11}/t & \underline{B}_{16}/t & \underline{D}_{11}/t^2 \end{vmatrix} \equiv \underline{D} \tag{14}$$

$$\underline{\varepsilon} = -(1/\underline{\Delta})\{\underline{A}_{16}(-\underline{S}+\underline{F}_{xy}) - \underline{A}_{66}(\underline{D\lambda}^2 t^2 + \underline{F}_x)\}, \tag{15a}$$

$$\underline{\gamma} = -(1/\underline{\Delta})\{\underline{A}_{11}(-\underline{S}+\underline{F}_{xy}) - \underline{A}_{16}(\underline{D\lambda}^2 t^2 + \underline{F}_x)\}, \tag{15b}$$

$$\underline{\xi} = (1/\underline{\Delta})(\underline{A}_{16}\underline{B}_{16}/t - \underline{A}_{66}\underline{B}_{11}/t)\theta\underline{\lambda}/(\sin\underline{\lambda}a), \tag{16a}$$

$$\underline{\eta} = (1/\underline{\Delta})(\underline{A}_{16}\underline{B}_{11}/t - \underline{A}_{11}\underline{B}_{16}/t)\theta\underline{\lambda}/(\sin\underline{\lambda}a), \tag{16b}$$

where $-\underline{P} \equiv \underline{N}_x$ and $\underline{S} \equiv \underline{N}_{xy}$ are the membrane forces and

$$\underline{\Delta} \equiv (\underline{A}_{11}\underline{A}_{66} - \underline{A}_{16}^2), \tag{17}$$

$$\theta \equiv (t/a)A\lambda \sin \lambda b. \tag{18}$$

$$\underline{F}_x \equiv \underline{N}_x^* + \underline{A}_{12}\beta + (\underline{B}_{11} + \underline{A}_{11}h/2)\kappa_x^0 + (\underline{B}_{12} + \underline{A}_{12}h/2)\kappa_y + (\underline{B}_{16} + \underline{A}_{16}h/2)\kappa_{xy}, \tag{19a}$$

$$\underline{F}_{xy} \equiv \underline{N}_{xy}^* + \underline{A}_{26}\beta + (\underline{B}_{16} + \underline{A}_{16}h/2)\kappa_x^0 + (\underline{B}_{26} + \underline{A}_{26}h/2)\kappa_y + (\underline{B}_{66} + \underline{A}_{66}h/2)\kappa_{xy}. \tag{19b}$$

Similar results for the upper disbonded sublaminate are obtained by replacing all underline quantities in eqns (14)–(17) by corresponding symbols with overhead bars, and replacing $\underline{F}_x$ and $\underline{F}_{xy}$, respectively, by the expressions

$$\bar{F}_x \equiv F_x - \underline{F}_x = \underline{N}_x^* + \bar{A}_{12}\beta + (\bar{B}_{11} - \bar{A}_{11}H/2)\kappa_x^0 + (\bar{B}_{12} - \bar{A}_{12}H/2)\kappa_y + (\bar{B}_{16} - \bar{A}_{16}H/2)\kappa_{xy}, \tag{20a}$$

$$\bar{F}_{xy} \equiv F_{xy} - \underline{F}_{xy} = \underline{N}_{xy}^* + \bar{A}_{26}\beta + (\bar{B}_{16} - \bar{A}_{16}H/2)\kappa_x^0 + (\bar{B}_{26} - \bar{A}_{26}H/2)\kappa_y + (\bar{B}_{66} - \bar{A}_{66}H/2)\kappa_{xy}. \tag{20b}$$

Substituting the preceding expressions of the deformation parameters into eqns (2) and (3), one obtains the deflections and the in-plane strains of the sublaminates. Equation

(7) and similar constitutive equations for the disbonded sublaminates then yield the sublaminate forces and moments. It may be verified, by using eqns (1) and (6), that the results satisfy *exactly* the equilibrium equations of the von Karman plate theory:

$$N_{\beta\gamma,\gamma} = \underline{N}_{\beta\gamma,\gamma} = \bar{N}_{\beta\gamma,\gamma} = 0, \quad (\beta = 1,2)$$
$$M_{\beta\gamma,\beta\gamma} - N_{\beta\gamma}w_{,\beta\gamma} = \underline{M}_{\beta\gamma,\beta\gamma} - \underline{N}_{\beta\gamma}\underline{w}_{,\beta\gamma} = \bar{M}_{\beta\gamma,\beta\gamma} - \bar{N}_{\beta\gamma}\bar{w}_{,\beta\gamma} = 0. \tag{21}$$

Consequently, for a given temperature load $T(z)$ and given mechanical load parameters β, κ_x^0, κ_y, κ_{xy}, S, and $P = D(\lambda t)^2$, solution of the postbuckling problem reduces to finding the deformation parameter θ and the membrane forces $\underline{S}$, $\bar{S} \equiv \bar{N}_{xy}$, $-\underline{P} = \underline{D}(\underline{\lambda} t)^2$ and $-\bar{P} \equiv \bar{N}_x = \bar{D}(\bar{\lambda} t)^2$ in the disbonded sublaminates.

These five unknowns may be solved from five algebraic relations resulting from the continuity of the in-plane displacements at the delamination front and the balance conditions of the axial forces, shearing forces and bending moments, i.e.,

$$\bar{u}(a,y) = \underline{u}(a,y) - (t/2)w_{,x}(a,y), \quad \bar{v}(a,y) = \underline{v}(a,y) - (t/2)a\kappa_{xy}, \tag{22a,b}$$

$$P = \underline{P} + \bar{P}, \quad S = \underline{S} + \bar{S}, \quad M_x(a) = \underline{M}_x(a) + \bar{M}_x(a) - \underline{P}h/2 + \bar{P}H/2. \tag{22c,d,e}$$

By further eliminating the two parameters $\underline{S}$ and $\bar{S}$ using eqns (22b,d) and the constitutive equations of the sublaminates, the set of unknown parameters is finally reduced to θ, $\underline{P} \equiv \underline{D}(\underline{\lambda} t)^2$ and $\bar{P} \equiv \bar{D}(\bar{\lambda} t)^2$. The system of equations governing these three unknown parameters is obtained from eqns (22a,c,e) by expressing all other parameters in terms of θ, $\underline{\lambda}$, $\bar{\lambda}$ and the mechanical and temperature loads. After lengthy algebraic manipulations and using the strain–displacement relation of eqn (4), the governing equations are reduced to the following explicit forms:

$$\underline{D}(\underline{\lambda} a)^2 + \bar{D}(\bar{\lambda} a)^2 = D(\lambda a)^2 \tag{23}$$

$$(\theta a^2/t^2)^2 \Gamma\{csc^2\bar{\lambda}a - (ctn\bar{\lambda}a)/\bar{\lambda}a - csc^2\underline{\lambda}a + (ctn\underline{\lambda}a)/\underline{\lambda}a\}$$
$$-4(\theta a^2/t^2)\begin{vmatrix} \underline{A}_{11} & \underline{A}_{16} & (\underline{B}_{11}+\underline{A}_{11}h/2)/t \\ A_{11} & A_{16} & B_{11}/t \\ A_{16} & A_{66} & B_{16}/t \end{vmatrix} - (4a^2/t^2)\begin{vmatrix} \underline{A}_{11} & \underline{A}_{16} & \underline{D}(\underline{\lambda} t)^2 + \underline{F}_x \\ \bar{A}_{11} & \bar{A}_{16} & \bar{D}(\bar{\lambda} t)^2 + \bar{F}_x \\ A_{16} & A_{66} & -S + F_{xy} \end{vmatrix} = 0. \tag{24}$$

$$(\theta a^2/t^2)\left\{\underline{\Delta}\bar{\Delta}\left(\frac{1}{\underline{\Delta}}\begin{vmatrix} \underline{A}_{11} & \underline{B}_{11}/t \\ \underline{A}_{16} & \underline{B}_{16}/t \end{vmatrix} - \frac{1}{\bar{\Delta}}\begin{vmatrix} \bar{A}_{11} & \bar{B}_{11}/t \\ \bar{A}_{16} & \bar{B}_{16}/t \end{vmatrix}\right)^2 \right.$$
$$\left. + \Gamma(D\lambda a\, ctn\lambda(L-a) + \underline{D}\underline{\lambda} a\, ctn\underline{\lambda} a + \bar{D}\bar{\lambda} a\, ctn\bar{\lambda} a)\right\}$$
$$+\begin{vmatrix} \underline{A}_{11} - \Gamma/\Delta & \underline{A}_{11} & \underline{A}_{16} & \underline{D}(\underline{\lambda} t)^2 + \underline{F}_x \\ \Gamma/\Delta & \bar{A}_{11} & \bar{A}_{16} & \bar{D}(\bar{\lambda} t)^2 + \bar{F}_x \\ \underline{A}_{16} & A_{16} & A_{66} & -S + F_{xy} \\ \underline{B}_{11}/t + \underline{A}_{11}h/2t & B_{11}/t & B_{16}/t & 0 \end{vmatrix} = 0, \tag{25}$$

where

$$\Gamma \equiv \underline{A}_{11}\bar{\Delta}+\bar{A}_{11}\underline{\Delta}.$$

When β, κ_x^0, κ_y, κ_{xy} and the temperature load are absent, eqns (24) and (25) reduce, respectively, to eqns (31) and (32) in Yin (1986).

Buckling and postbuckling problems of homogeneous delaminated plates under compression and bending loads were first formulated by Kachanov (1976), who also suggested a delamination growth criterion based on the strain energy release rate. Plane strain, cylindrical postbuckling deformation accounting for the interaction of the intact and disbonded segments of a homogeneous plate was formulated and solved by Chai *et al.* (1981) and, for delaminated cylindrical shells, by Troshin (1982). An analysis that showed two different types of postbuckling behavior (depending on the length and the thickness ratios of the sublaminate vs the based plate) with important implications on the delamination growth process was given by Yin *et al.* (1986). The analysis was extended to multilayered delamination models with arbitrary ply configurations, and exact solutions were obtained by allowing the membrane strains to depend sinusoidally on the axial coordinate (Yin, 1986; 1988). Although such sinusoidal variation of the membrace strains and displacements is a prominent feature of the solutions of strip delamination models with bending–stretching coupling, it has not been widely recognized. Previous analytical studies which failed to include this feature might not yield correct solutions for the buckling load and the postbuckling deformation. Further generalization in the analysis of strip delamination models was made by Chen (1991) to include the effects of out-of-plane shear deformation. For anisotropic delamination models subjected to temperature loads, the usual assumption of cylindrical deformation is no longer valid, and the present formulation is needed to account for biaxial bending and twisting of the sublaminates.

We notice that the temperature load $T(z)$ affects the preceding governing equations and the postbuckling solution only through the thermal forces $\underline{N}_x^*$, $\underline{N}_{xy}^*$, $\bar{N}_x^*$ and $\bar{N}_{xy}^*$, and these effective forces appear only through $\underline{F}_x$, $\underline{F}_{xy}$, $\bar{F}_x$ and $\bar{F}_{xy}$ as defined, respectively, by eqns (19) and (20). Given any temperature load $T(z)$, one can generally replace it with equivalent increments of the strain and curvature loads, $\Delta\beta$, $\Delta\kappa_x^0$, $\Delta\kappa_y$ and $\Delta\kappa_{xy}$, such that the resulting postbuckling solution remains unchanged. These equivalent increments in β, κ_x^0, κ_y, κ_{xy} are given by the solutions of the linear equations

$$\begin{aligned}
\underline{A}_{12}\,\Delta\beta+(\underline{B}_{11}-\underline{A}_{11}H/2)\,\Delta\kappa_x^0+(\underline{B}_{12}+\underline{A}_{12}h/2)\,\Delta\kappa_y+(\underline{B}_{16}+\underline{A}_{16}h/2)\,\Delta\kappa_{xy} &= \underline{N}_x^*,\\
\underline{A}_{26}\,\Delta\beta+(\underline{B}_{16}-\underline{A}_{16}H/2)\,\Delta\kappa_x^0+(\underline{B}_{26}+\underline{A}_{26}h/2)\,\Delta\kappa_y+(\underline{B}_{66}+\underline{A}_{66}h/2)\,\Delta\kappa_{xy} &= \underline{N}_{xy}^*,\\
\bar{A}_{12}\,\Delta\beta+(\bar{B}_{11}-\bar{A}_{11}H/2)\,\Delta\kappa_x^0+(\bar{B}_{12}-\bar{A}_{12}H/2)\,\Delta\kappa_y+(\bar{B}_{16}-\bar{A}_{16}H/2)\,\Delta\kappa_{xy} &= \bar{N}_x^*,\\
\bar{A}_{26}\,\Delta\beta+(\bar{B}_{16}-\bar{A}_{16}H/2)\,\Delta\kappa_x^0+(\bar{B}_{26}-\bar{A}_{26}H/2)\,\Delta\kappa_y+(\bar{B}_{66}-\bar{A}_{66}H/2)\,\Delta\kappa_{xy} &= \bar{N}_{xy}^*.
\end{aligned}$$

Thus, in general, the thermoelastic postbuckling problem of the strip delamination model may be reduced to a purely mechanical problem when the original strain and curvature loads β, κ_x^0, κ_y and κ_{xy} are replaced by $\beta+\Delta\beta$, $\kappa_x^0+\Delta\kappa_x^0$, $\kappa_y+\Delta\kappa_y$ and $\kappa_{xy}+\Delta\kappa_{xy}$, respectively. Exceptions to this statement occur in some degenerate cases, including homogeneous isotropic delaminated plates, when the preceding equations for $\Delta\beta$, $\Delta\kappa_x^0$, $\Delta\kappa_y$ and $\Delta\kappa_{xy}$ have no solutions due to the singularity of the coefficient matrix.

For the postbuckling solutions of the present analysis, the boundary value of the axial moment, $M = M_x(L)$, is determined by the thermal and mechanical loads and the deflection amplitude A according to the fourth row of eqn (7). One has

$$\begin{vmatrix} A_{11} & A_{16} & D\lambda^2 t^2+F_x \\ A_{16} & A_{66} & -S+F_{xy} \\ B_{11}/t & B_{16}/t & -M+M_x^*+B_{12}\beta+D_{11}\kappa_x^0+D_{12}\kappa_y+D_{16}\kappa_{xy} \end{vmatrix} = D\,\Delta\theta\lambda a/(\sin\lambda b). \quad (26)$$

Hence, in a slightly different formulation of the postbuckling problem, where β, κ_y, κ_{xy}, λ,

S and M (instead of β, κ_y, κ_{xy}, λ, S and κ_x^0) are taken to be specified mechanical load parameters accompanying the temperature load, the reduced set of governing algebraic equations, eqns (23)–(25), must be augmented by eqn (26) to determine a set of four deformation parameters θ, $\underline{\lambda}$, $\bar{\lambda}$ and κ_x^0. Notice, that in this case, the final expression for the postbuckling solution will involve the thermal moment M_x^* because eqn (26) contains the combination $-M+M_x^*$.

5. POSTBUCKLING DEFORMATION AND THE ENERGY RELEASE RATE

The separation of the two disbonded sublaminates at $x = 0$ is given by

$$\bar{w}(0)-\underline{w}(0) = t(\theta a^2/t^2)\{\tan(\underline{\lambda}a/2)/(\underline{\lambda}a)-\tan(\bar{\lambda}a/2)/(\bar{\lambda}a)\}. \tag{27}$$

This relation is valid regardless of the temperature load. Notice that eqn (44) of Yin (1986) is incorrect and should be replaced by the preceding expression. The deflections given by eqns (2b) and (2c) are physically possible only if $\bar{\kappa}_x \geqslant \underline{\kappa}_x$ at $x = a$. Otherwise the actual postbuckling solution should be obtained by considering contact of the upper and lower disbonded sublaminates in an interval adjacent to the crack tip. For given mechanical and thermal loads P, S, β, κ_x^0, κ_y, κ_{xy} and $T(z)$, eqns (23)–(25) may be solved for the reduced set of unknown parameters θ, $\underline{\lambda}$ and $\bar{\lambda}$. Equations (2a,b,c) and (18) then yield the deflection functions of the sublaminates. All parameters in the expressions of the membrane strains, eqns (3a–i), are given by eqns (9), (10), (15), (16) and by similar equations of the upper sublaminate.

Under sufficiently large mechanical and thermal loads, the force and moment resultants at the crack tip may cause severe interlaminar peeling and shearing actions ahead of the delamination crack and may initiate delamination growth. An important measure of the crack driving force is the energy release rate G. On the basis of the laminated plate theory, G may be expressed in terms of the midplane strains and the curvatures of the sublaminates at the crack tip. The expression may be derived either by considering local energy balance as the crack advances an infinitesimal distance, or by evaluating the path-independent J-integral (Yin and Wang, 1984; Yin, 1986). The result is presented here in a concise form:

$$G = 1/2\{\Delta\bar{\varepsilon}\}^T\begin{bmatrix}\bar{A}_{11} & \bar{A}_{16} & \bar{B}_{11}\\ \bar{A}_{16} & \bar{A}_{66} & \bar{B}_{16}\\ \bar{A}_{11} & \bar{A}_{16} & \bar{D}_{11}\end{bmatrix}\{\Delta\bar{\varepsilon}\}+1/2\{\Delta\underline{\varepsilon}\}^T\begin{bmatrix}\underline{A}_{11} & \underline{A}_{16} & \underline{B}_{11}\\ \underline{A}_{16} & \underline{A}_{66} & \underline{B}_{16}\\ \underline{B}_{11} & \underline{B}_{16} & \underline{D}_{11}\end{bmatrix}\{\Delta\underline{\varepsilon}\}, \tag{28}$$

where

$$\{\Delta\bar{\varepsilon}\}^T \equiv \{\bar{\varepsilon}_x^0-(\varepsilon_x^0-\kappa_x H/2), \bar{\gamma}_{xy}^0-(\gamma_{xy}^0-\kappa_{xy}H/2), \bar{\kappa}_x-\kappa_x\},$$

$$\{\Delta\underline{\varepsilon}\}^T \equiv \{\underline{\varepsilon}_x^0-(\varepsilon_x^0+\kappa_x h/2), \underline{\gamma}_{xy}^0-(\gamma_{xy}^0+\kappa_{xy}h/2), \underline{\kappa}_x-\kappa_x\},$$

are evaluated at the crack tip and where $\{\Delta\bar{\varepsilon}\}$ and $\{\Delta\underline{\varepsilon}\}$ are the corresponding column vectors. Notice that this expression does not explicitly involve β and κ_y. Alternatively, G may be given in terms of $N_{\alpha\beta}-N_{\alpha\beta}^*$, $M_{\alpha\beta}-M_{\alpha\beta}^*$, $\underline{N}_{\alpha\beta}-\underline{N}_{\alpha\beta}^*$, $\underline{M}_{\alpha\beta}-\underline{M}_{\alpha\beta}^*$, $\bar{N}_{\alpha\beta}-\bar{N}_{\alpha\beta}^*$ and $\bar{M}_{\alpha\beta}-\bar{M}_{\alpha\beta}^*$ by using eqn (7) and the corresponding constitutive equations for the disbonded sublaminates (Yin, 1998).

We note that eqn (28) applies to a 2-D delamination of arbitrary shape if the x- and y-axes (i.e., the referential axes for the components of the mid-plane strains and curvatures and for the various stiffness matrices of the intact and delaminated sublaminates) are replaced, respectively, by the in-plane local axes normal and tangential to the delamination front (Yin, 1996). This expression, given in terms of the local strains and curvatures, remains valid when the mechanical load is accompanied by a general temperature load $T(x, y, z)$, provided that the latter is continuous across the delamination front. This is because, as the crack front advances by an infinitesimal distance Δa, there are sudden changes in the strains and curvatures as the intact segment splits in the process zone but there is no sudden change in the potential energy associated with the thermal strain if the temperature field is continuous. Consequently, eqn (28) remains valid regardless of the presence or absence of the thermal strain. On the other hand, the alternative form of G in terms of the stress and moment resultants in the sublaminates (or a mixed expression in terms of both kinetic and kinematic quantities of the sublaminates) will involve both mechanical and thermal forces.

For a delamination crack between two layers of identical material with the same orientation, the energy release rate may be separated into three components associated, respectively, with the peeling action and the two modes of out-of-plane shear action. Delamination growth criteria that discriminate among the three distinct fracture modes may be formulated in terms of the three modal components of G. However, modal separation of G cannot be achieved within the context of the laminated plate theory; it requires a asymptotic elasticity solution near the crack tip for the determination of the ratio of the corresponding stress intensity factors. For interface cracks between two dissimilar elastic layers, the energy release rate generally cannot be partitioned into separate modal components. There are no states of loading that yield pure mode I or mode II behavior at the crack tip, and the shearing and peeling actions are intrinsically intertwined in the asymptotic solution. An analysis procedure for accurately determining the local elasticity solution along the crack front of a general 2-D delamination has been developed elsewhere (Yin, 1996; 1997), based on the computed sublaminate forces and moments which provide boundary traction data along a small circular path that encircles the crack tip. The elasticity solution is given in terms of a complex eigenfunction series and the dominant singular term of the series determines the asymptotic solution and the complex stress intensity factor.

6. RESULTS

We consider delaminations in multilayered B(4)/5505 boron/epoxy laminates and in homogeneous isotropic plates with a moderate expansion coefficient. For the boron/epoxy composite the layer properties are taken from Tsai and Hahn (1980), i.e., $E_1 = 204$ GPa, $E_2 = 18.5$ GPa, $G_{12} = 5.59$ GPa, $v = 0.23$, $\alpha_1 = 6.1\times10^{-6}/^\circ$K and $\alpha_2 = 30.3\times10^{-6}/^\circ$K. In order that results comparable in magnitude may be obtained for the homogeneous delaminated plates, the isotropic material is assumed to have a thermal expansion coefficient $10.0\times10^{-6}/^\circ$K and Poisson's ratio 0.3. The bifurcation loads of strip delamination models made of such materials have been determined by Yin (1994).

Although β, κ_x^0, κ_y, κ_{xy} and the in-plane shearing force S generally affect the postbuckling solution, the essential features of postbuckling response may be found by concentrating on the effects of axial compression and the temperature load. Therefore, postbuckling solutions will be computed in the following analysis under the assumption that $\beta = \kappa_x^0 = \kappa_y = \kappa_{xy} = S = 0$. To be specific, we also assume that the temperature load varies linearly across the thickness, and assumes the values $T_L = -T_U$, 0 and T_U, respectively, on $z = -t/2$, 0 and $t/2$. We note that, while a uniform temperature load does not affect the postbuckling deflection of a homogeneous delamination model under a force-controlled axial compression P, it generally changes the solution of a multilayered delamination model because of the presence of the thermal forces in eqns (24) and (25).

It is convenient to normalize the axial load P with respect to the axial bifurcation load of an otherwise identical clamped laminate without delamination, $P_{cr}^0 = (\pi t/L)^2 D$. This yields

$$\mathscr{P} \equiv P/P_{cr}^0 = (P/D)(L/\pi t)^2. \tag{29}$$

For the postbuckling solutions of a homogeneous isotropic plate with a moderately deep

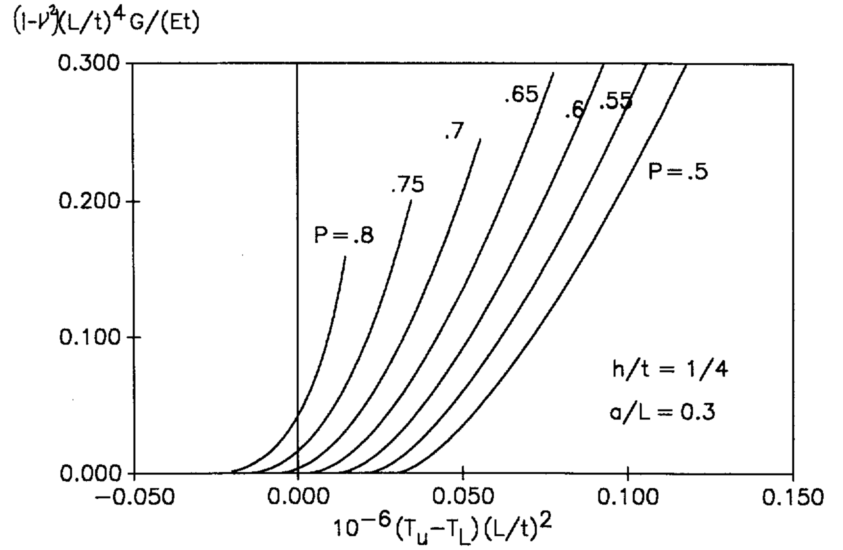

Fig. 2a. Postbuckling energy release rate for a delamination in a homogeneous isotropic plate, $h/t = 1/4$, $a/L = 0.3$.

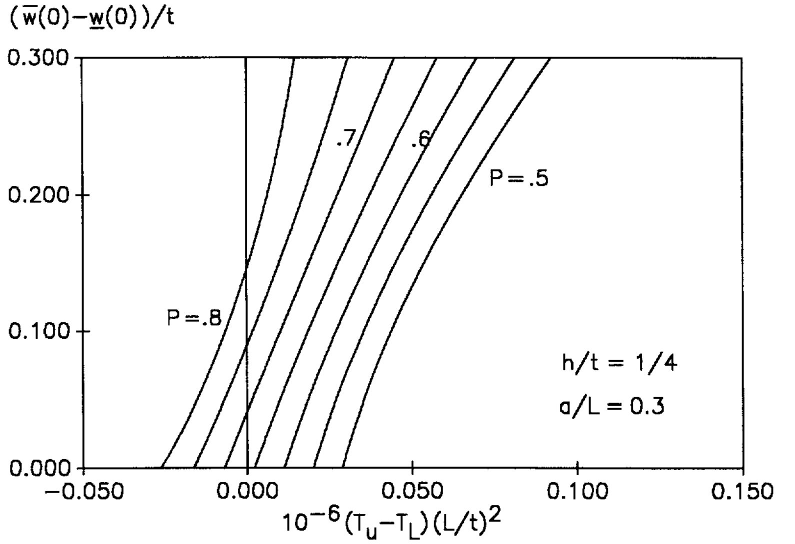

Fig. 2b. Midpoint separation. Delamination in a homogeneous isotropic plate, $h/t = 1/4$, $a/L = 0.3$.

delamination ($h/t = 1/4$ and $a/L = 0.3$), the energy release rates and the midpoint separations are shown in Figs 2a and b, respectively, for seven values of $\mathscr{P}$ and for increasing values of the temperature gradient. The corresponding results for a relatively shallow and short delamination (with $h/t = 1/8$ and $a/L = 0.15$) are shown in Figs 3a and b.

Significant amounts of midpoint separation and energy release rate generally occur in a relatively advanced stage of postbuckling deformation. In this stage one may find a simple approximate relation between the energy release rate and the midpoint separation. Let the separation be normalized with respect to h (rather than the laminate thickness t), and let G be normalized with respect to $(h/a)^4$ times the axial extensional stiffness of the thinner disbonded sublaminate (rather than $(t/L)^4$ times the axial stiffness of the intact sublaminate). Then, for the new normalized variables $\hat{G}$ and δ, the following approximate relation is valid:

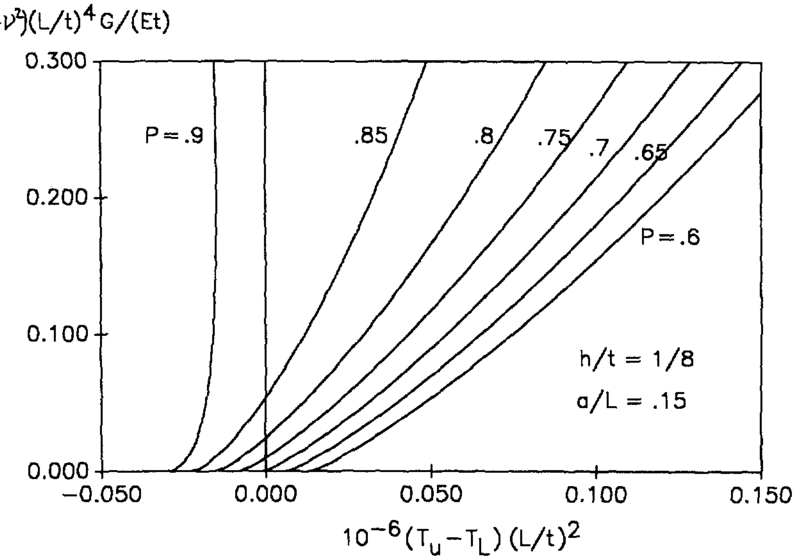

Fig. 3a. Postbuckling energy release rate for a delamination in a homogeneous isotropic plate, $h/t = 1/8$, $a/L = 0.15$.

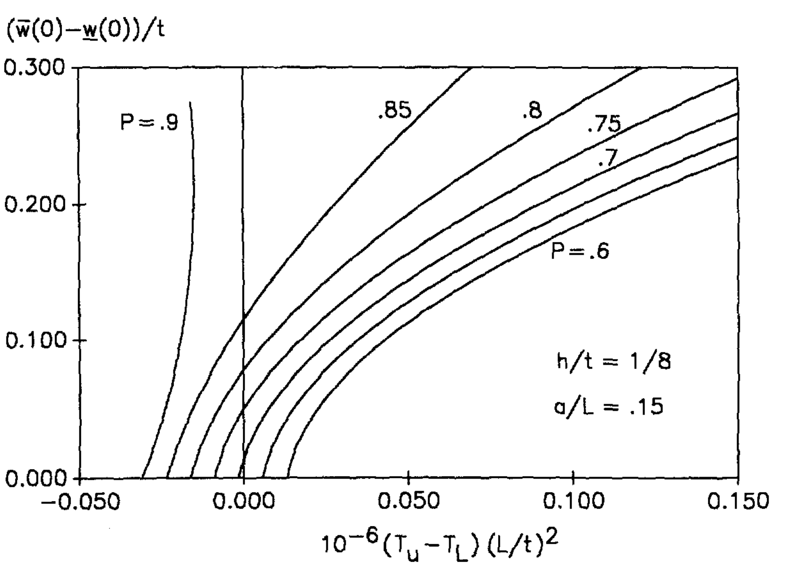

Fig. 3b. Midpoint separation. Delamination in a homogeneous isotropic plate, $h/t = 1/8$, $a/L = 0.15$.

$$2(4/\pi)^4\hat{G} \approx \delta^2\{\delta^2+64D/(A_{11}h)\} \tag{30}$$

where

$$\hat{G} \equiv (a/h)^4 G/A_{11}, \quad \delta \equiv \{\bar{w}(0)-\underline{w}(0)\}/h.$$

This exceedingly simple relation holds exactly in the limit of a thin-film delamination mode, and may be easily derived from eqns (11) and (16) of Yin (1988) when the temperature load and the bending and twisting curvatures are absent. A more general derivation shows that, with the inclusion of the temperature and curvature loads, eqn (30) is still valid in the limit of a thin-film delamination.

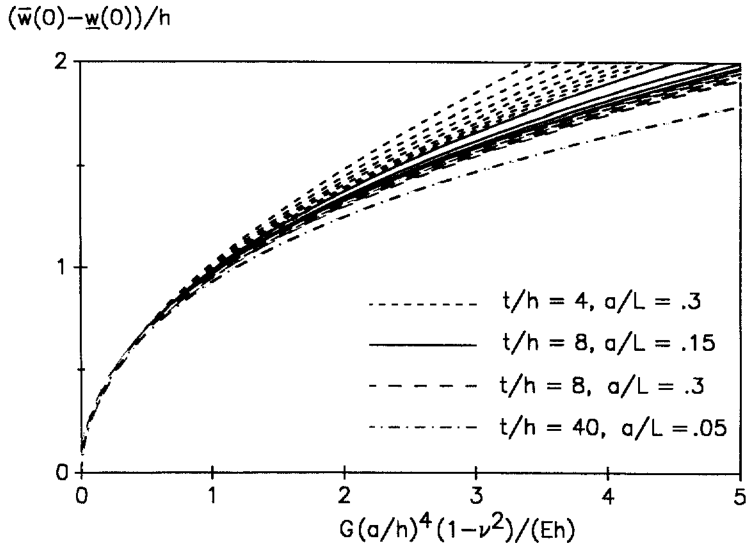

Fig. 4. Midpoint separation vs normalized energy release rate, delamination models with various thickness and length ratios.

The results of Figs 2a, b and 3a, b, and additional results corresponding to two other combinations of the thickness and length ratios are presented in Fig. 4 in terms of the new normalized quantities, $\hat{G}$ and $\hat{\delta}$. Deviations of the curves from the relation of eqn (30) (shown by the lowest curve in Fig. 4) indicate the errors associated with the thin-film approximation. It is found that, for a very thin delamination with $h/t = 1/40$ and $a/L = 1/20$, the results agree very closely with eqn (30). For other combinations of the thickness and length ratios, the agreement is sufficiently close whenever the midpoint separation is smaller than the thickness of the upper disbonded sublaminate. But if the latter condition is not satisfied, then the accuracy of the present postbuckling solutions (which are based on the von Karman theory of plates) becomes doubtful as well. In all cases, the results of Fig. 4 suggest that eqn (30) overestimates the energy release rate for a given midpoint separation. Hence if h and the midpoint separation are measured experimentally and if G is estimated by using eqn (30), a conservative prediction of delamination fracture can be obtained and the estimation of the energy release rate is close if the measured separation is comparable to or smaller than h.

For symmetric, eight-layer, cross-ply boron/epoxy laminates with a delamination on the second highest and the highest interface, respectively, the postbuckling energy release rates are shown in Figs 5 and 6 for several values of the axial load and as functions of the temperature gradient. The corresponding results for delaminated 45° angle-ply laminates are shown in Figs 7 and 8. Since the thermal expansion coefficients of the unidirectional boron/epoxy composite have a ratio of nearly 5:1 in the transverse direction compared to the fiber direction, the angle-play configuration is more prone to the destabilizing effect of a temperature gradient. Hence the curves in Figs 7 and 8 are steeper than those of Figs 5 and 6.

Consider a delaminated plate with $L/t = 25$ subjected to a temperature load with $T_U - T_L = 80°K$. This yields $10^{-6}(T_U - T_L)(L/t)^2 = 0.05$. Under a given mechanical load P, the imposition of this modest temperature load may significantly and even drastically increase the midpoint separation and the energy release rate in the postbuckling regime. This may be seen, for example, by comparing the values of G for each curve in Fig. 5 (or the values of the midpoint separation for each curve in Fig. 2b) at the two points with horizontal coordinates 0 and 0.05. Because the curves associated with greater $\mathscr{P}$ have larger slopes, they indicate greater effects of the temperature load upon the postbuckling deformation and the energy release rate.

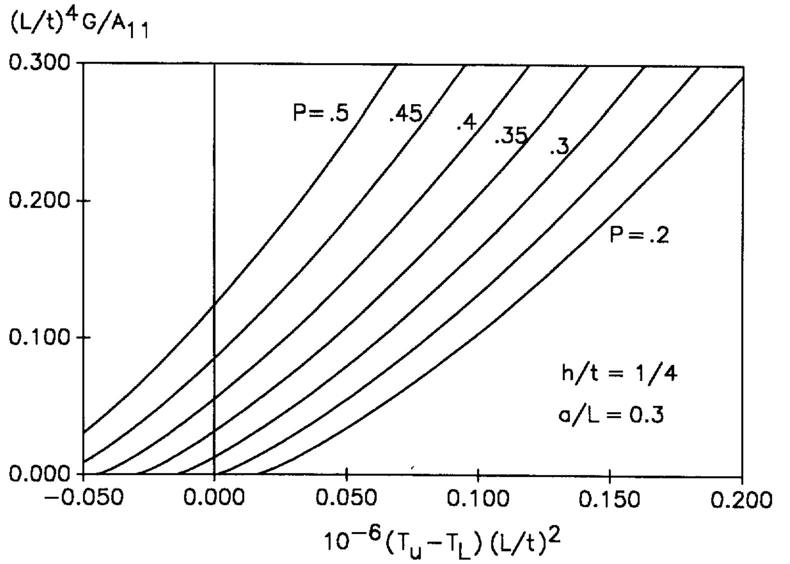

Fig. 5. Postbuckling energy release rate. Delamination in the second interface from the top of a $[(0/90)_2]_s$ boron/epoxy laminate.

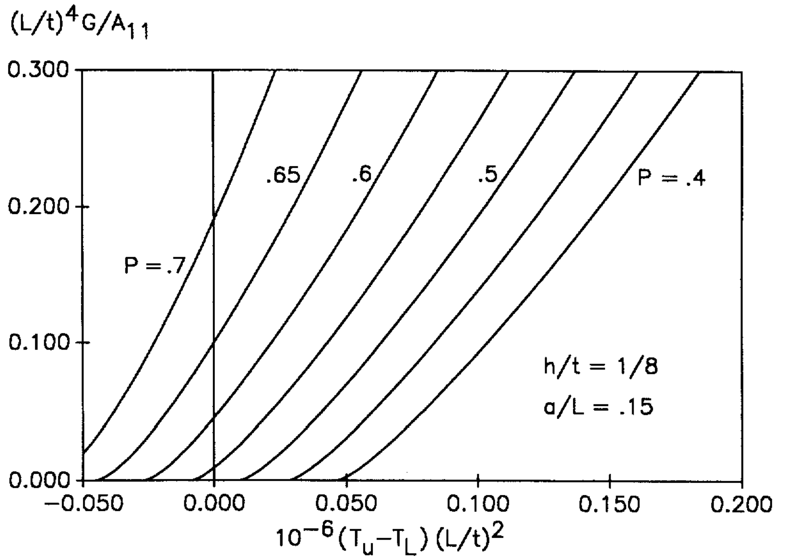

Fig. 6. Postbuckling energy release rate. Delamination in the first interface from the top of a $[(0/90)_2]_s$ boron/epoxy laminate.

7. SUMMARY AND CONCLUDING REMARKS

Postbuckling solutions of anisotropic strip delamination models under combined mechanical and temperature loads have been obtained on the basis of the laminated plate theory and under the assumption of generalized 2-D deformation. The model may be subjected to biaxial bending and twisting, in addition to in-plane compression and shearing forces. In the absence of transverse bending and twisting curvatures, the present postbuckling solutions are exact in the context of the laminated plate theory. Due to bending–extension coupling, the middle-plane strains of the intact and disbonded sublaminates vary sinusoidally with the axial coordinate, even though the in-plane forces are constant in each sublaminate. The effects of the temperature field on the buckling load and the postbuckling deformation are completely characterized by the thermal forces in the disbonded sublaminates, i.e., $\underline{N}_x^*$, $\underline{N}_{xy}^*$, $\bar{N}_x^*$ and $\bar{N}_{xy}^*$.

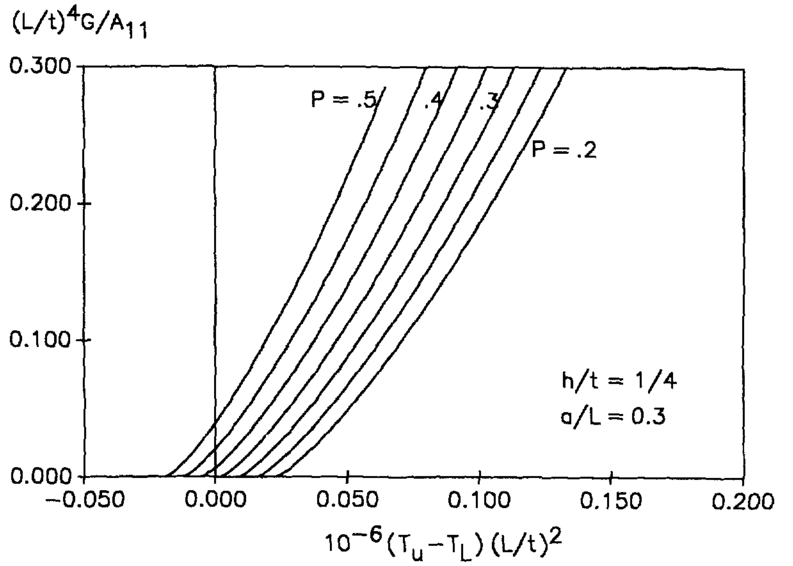

Fig. 7. Postbuckling energy release rate. Delamination in the second interface from the top of a $[(45/-45)_2]_s$ boron/epoxy laminate.

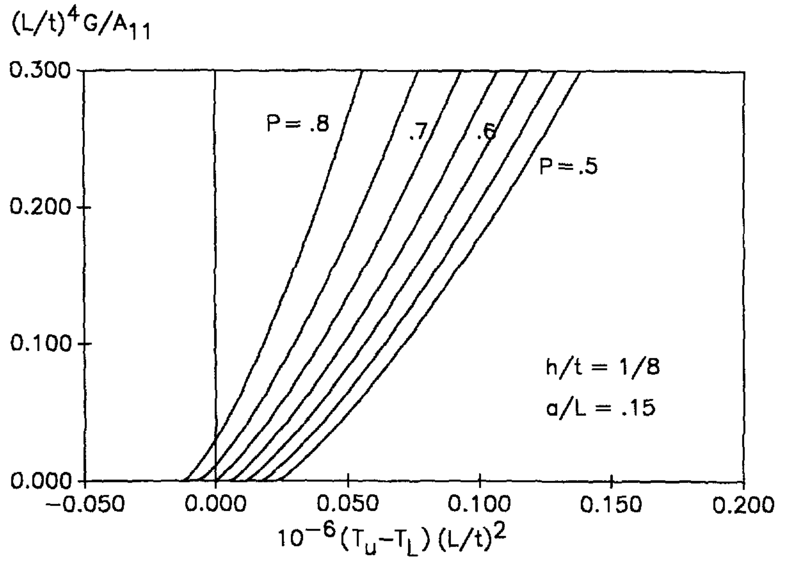

Fig. 8. Postbuckling energy release rate. Delamination in the first interface from the top of a $[(45/-45)_2]_s$ boron/epoxy laminate.

The three equations (23)–(25) governing the reduced set of deformation parameters θ, $\underline{\lambda}$ and $\bar{\lambda}$ show explicitly the effects of mechanical and temperature loads and of the stiffness and geometrical parameters on postbuckling behavior. Because the delamination model consists of multilayered sublaminates with arbitrary ply configurations and because the analysis encompasses thermal and mechanical loads including biaxial bending and twisting, a very large class of analytical solutions may be easily obtained by solving the three algebraic equations. The various parameters regarding geometry, material properties and loading may be selected so that the model may closely approximate composite beams and panels in aerospace applications and may provide useful results for assessing damage tolerance of the components.

Analysis results show that a temperature gradient in the thickness direction may significantly increase the midpoint separation of the disbonded sublaminates and the energy release rate associated with postbuckling delamination growth. Furthermore, the greater the axial compression force, the more threatening is the aggravating effect of the temperature load.

A general expression of the total energy release rate is given by eqn (28) in terms of the local membrane strains and curvatures of the sublaminates at the delamination front. The validity of this expression is unaffected by a temperature load that accompanies the mechanical load. In the limiting case of thin-film strip delamination, the expression reduces to an exceedingly simple relation, eqn (30), between the normalized energy release rate and the normalized midpoint separation δ. This relation is useful in estimating the threat of delamination damage because δ may be found by direct measurement. The present solutions all indicate that, for an observed separation, the use of the approximate relation of eqn (30) in a delamination growth criterion yields conservative prediction of growth initiation because eqn (30) yields a greater result of G than does eqn (28).

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